Consistency of Compact MOSFET Models with the Pao-Sah Formulation: Consequences for Small-Signal Analysis and Design

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ABSTRACT

Compact models for the MOSFET are based on the decomposition of the two-dimensional problem into two one-dimensional problems. Since a compact MOSFET model core consists of an input voltage equation, and an output current equation, a consistent compact model must approximate these two (orthogonal) equations consistently. In this study we will review some of the existing compact models, beginning with the Pao-Sah formula and including classical strong inversion, surface potential, and charge control models. A simple consistency test is applied to some compact models regarding the small-signal output conductance of the transistor.

Keywords: MOS transistor, MOSFET models, compact models

1 INTRODUCTION

Compact models for the MOSFET are based on the decomposition of the two (three)-dimensional problem into two (three) one-dimensional problems (1-D) [1]. For a long-channel device, the gradual channel approximation is valid, i.e., the longitudinal (y-direction) component of the electric field can be assumed to be much smaller than the transversal (x-direction) component. The 1-D x-equation, which is known as the input voltage equation, relates the applied gate voltage to the electric conditions of the semiconductor surface. The 1-D y-equation, which is known as the output current equation, relates the current flowing through drain and source to the x-solution and the voltages applied to the source and drain.

Because a compact MOSFET model core consists of an input voltage equation and an output current equation, a consistent compact model must approximate these two (orthogonal) equations consistently. In this study we will review some compact models, including the classical strong inversion, surface potential and charge control models and check their consistency with the Pao-Sah formula, which is highly physical and has served as a reference to test the accuracy of compact models [1]. The consistency test to be applied in this paper consists in verifying whether the (small-signal) output conductance of the MOSFET calculated with the inversion charge density at the drain end of the channel coincides with that calculated from the derivative of the drain current with respect to the drain voltage. The (non) coincidence of the results attests to the (non) consistency of the one-dimensional x-equation with the y-equation.

This simple consistency test regarding the (small-signal) output conductance of the transistor is applied to some popular MOSFET models.

2 CONSISTENCY TEST

The Pao-Sah current equation [2] is

$$I_D = -\mu W Q_I' \frac{dV_C}{dy} \tag{1}$$

where W is the channel width, μ is the carrier mobility, Q_I' is the inversion charge density and V_C is the channel potential (quasi-Fermi potential splitting). Expression (1) includes both the drift and diffusion transport mechanisms, and gives an exact model of the long-channel MOSFET. For this reason Eq. (1) is used as a golden reference to test the accuracy of compact models.

Since the current is constant along the channel, the integration of Eq. (1) from source to drain, assuming constant mobility, yields

$$I_D = -\frac{\mu W}{L} \int_{V_S}^{V_D} Q_I' dV_C \tag{2}$$

where L is the channel length and V_D and V_S are the drain-to-bulk and source-to-bulk potentials, respectively.

Incremental (small-signal) parameters are essential for analog and RF design. Also, in the context of compact modeling, small-signal parameters of the MOSFET can serve to check the consistency of the one-dimensional x-equation with the y-equation. Since complete small-signal models of the MOSFET can be rather involved we will restrict the small-signal analysis to the calculation of the output conductance. A complete discussion on the remaining small-signal parameters is available in [3].

The small-signal output conductance of the transistor is defined as

$$g_d = \left.\frac{\partial I_D}{\partial V_D}\right|_{V_G, V_S} \tag{3}$$

where V_G *is* the gate-to-bulk potential.

Applying the definition of Eq. (3) to the Pao-Sah formula (1), we obtain the important result

$$g_d = -\mu \frac{W}{L} Q_I'(V_D) \tag{4}$$

A basic consistency test for compact models consists in comparing g_d calculated through (3) using the model particular y-equation to g_d evaluated through (4) using the respective x-equation. Clearly, all physically meaningful models derived from Pao-Sah expression (1) must satisfy Eq.(4).

3 REVIEW OF COMPACT MODELS

3.1 Surface potential models

The first integral of the Poisson (voltage input) equation can be written as [4]

$$(V_G - V_{FB} - \phi_s)^2 = \gamma^2 \left\{ \phi_t \left[e^{-\frac{\phi_s}{\phi_t}} + \frac{\phi_s}{\phi_t} - 1 \right] + \phi_t e^{-\frac{(2\phi_F + V_C)}{\phi_t}} \left[e^{\frac{\phi_s}{\phi_t}} - \frac{\phi_s}{\phi_t} - 1 - \chi(\phi_s) \right] \right\} \tag{5}$$

where V_{FB} is the flat-band voltage, ϕ_s is the surface potential, γ is the body effect factor, ϕ_t is the thermal voltage and ϕ_F is the body Fermi potential. The function $\chi(\phi_s)$ was defined in different ways [4] by different authors in order to circumvent some (numerical) problems of (5) near the flat-band condition. The simplest solution consists of choosing $\chi(\phi_s) = 0$.

In inversion ($\phi_s > \phi_F$) Eq. (5) can be approximated by

$$(V_G - V_{FB} - \phi_s)^2 = \gamma^2 \left(\phi_t e^{(\phi_s - 2\phi_F - V_C)/\phi_t} + \phi_s - \phi_t \right) \tag{6}$$

with high accuracy.

The voltage input equation (x-equation) of surface potential models is either Eq. (5) or Eq. (6). The current output equation (y-equation) is in general obtained using the *charge-sheet approximation* [5] to calculate the inversion charge density $Q_I'(\phi_s)$:

$$Q_I' = -C_{ox}' \left(V_G - V_{FB} - \phi_s - \gamma \sqrt{\phi_s - \phi_t} \right) \tag{7}$$

where C_{ox}' is the oxide capacitance per unit area. The surface potential model will fulfill the consistency test here presented if the drain conductance g_d calculated from (3) is equal to

$$g_d = \mu \frac{W}{L} C_{ox}' \left(V_G - V_{FB} - \phi_s - \gamma \sqrt{\phi_s - \phi_t} \right) \tag{8}$$

There are two main approaches for deriving the surface potential models y-equation:

a) Brews' approach

Brews' approach is the most well-known charge-sheet model, which can be derived [5] from the charge-sheet expression for the current, rewritten below

$$I_D = I_{drift} + I_{diff} = -\mu W Q_I' \frac{d\phi_s}{dy} + \mu W \phi_t \frac{dQ_I'}{dy} \tag{9}$$

Substituting (7) into (9) it follows that

$$I_{drift} = \mu \frac{W}{L} C_{ox}' \Big\{ (V_G - V_{FB})(\phi_{sL} - \phi_{s0}) - \frac{1}{2}\left(\phi_{sL}^2 - \phi_{s0}^2\right) - \frac{2}{3}\gamma \left[(\phi_{sL} - \phi_t)^{3/2} - (\phi_{s0} - \phi_t)^{3/2} \right] \Big\} \tag{10.a}$$

$$I_{diff} = \mu \frac{W}{L} C_{ox}' \phi_t \left\{ (\phi_{sL} - \phi_{s0}) + \gamma \left[(\phi_{sL} - \phi_t)^{1/2} - (\phi_{s0} - \phi_t)^{1/2} \right] \right\} \tag{10.b}$$

where ϕ_{s0} and ϕ_{sL} are the values of the surface potential at the source and drain channel ends, respectively.

Applying the definition in (3) to (9) and (10), the output conductance of Brews' model is given by

$$g_d = \mu \frac{W}{L} C_{ox}' \frac{V_G - V_{FB} - \phi_{sL} - \gamma \sqrt{\phi_{sL} - \phi_t} + \phi_t \left(1 + \dfrac{\gamma}{2\sqrt{\phi_{sL} - \phi_t}} \right)}{\left.\dfrac{\partial V_D}{\partial \phi_{sL}}\right|_{V_G}} \tag{11}$$

where the denominator in the right-hand side is equal to $\left.\frac{\partial V_C}{\partial \phi_s}\right|_{V_G}$ with $V_C = V_D$ and $\phi_s = \phi_{sL}$.

Differentiating Eq. (6) with respect to ϕ_s we obtain [6, 7]

$$\frac{dV_C}{d\phi_s} = 1 + \phi_t \frac{2(V_G - V_{FB} - \phi_s) + \gamma^2}{(V_G - V_{FB} - \phi_s)^2 - \gamma^2(\phi_s - \phi_t)} \tag{12}$$

Equation (12) gives a very accurate value of $dV_C/d\phi_s$ in inversion. The "exact" expression of $dV_C/d\phi_s$ determined from Eq. (5) contains exponential terms and does not allow the obtention of a closed expression for the drain current as already observed by Van de Wiele [7].

We notice, however that, since the inversion charge density is given by (7), in order to satisfy the consistency test of (4), the denominator of (11) should be calculated from

$$\left.\frac{\partial V_C}{\partial \phi_s}\right|_{V_G} = 1 + \frac{\phi_t}{V_G - V_{FB} - \phi_s - \gamma\sqrt{\phi_s - \phi_t}}\left(1 + \frac{\gamma}{2\sqrt{\phi_s - \phi_t}}\right) \tag{13}$$

Equation (13) is only an approximation of the very accurate Eq. (12); however, Eq. (13) is the only one which satisfies simultaneously the Brews' charge-sheet model and the Pao-Sah equation. Therefore, the pair of orthogonal equations (6) and (10) does not pass the consistency test. It has been shown [7, 8] that approximation (13) along with the Pao-Sah current expression allows one to derive expression (9) of the charge-sheet current.

b) Approximation of the Pao-Sah integral

To obtain compact approximations for the drain current which are valid in the different inversion regimes, Pao-Sah integral (2) can be calculated by means of a change of variable and appropriate approximations. Eq.(2) is rewritten in the form

$$I_D = -\mu \frac{W}{L}\int_{\phi_{s0}}^{\phi_{sL}} Q_I'(\phi_s)\frac{dV_C}{d\phi_s}d\phi_s \tag{14}$$

where the integration is carried out over variable ϕ_s instead of V_C.

To calculate the integral in (14), we can use the *charge-sheet approximation* of $Q_I'(\phi_s)$, Eq.(7), together with an approximation for $dV_C/d\phi_s$. Therefore, Eq. (4) will only occasionally be satisfied, being thus a first consistency test for models derived as approximations of the Pao-Sah integral.

For instance, substituting expressions (7) and (12) into Eq. (14) we obtain one of the first surface-potential models [6] published

$$I_D = \mu \frac{W}{L}C_{ox}'\int_{\phi_{s0}}^{\phi_{sL}}\left(V_G - V_{FB} - \phi_s - \gamma\sqrt{\phi_s - \phi_t}\right)d\phi_s + \\ + \mu\frac{W}{L}C_{ox}'\phi_t\int_{\phi_{s0}}^{\phi_{sL}}\frac{2(V_G - V_{FB} - \phi_s) + \gamma^2}{V_G - V_{FB} - \phi_s - \gamma\sqrt{\phi_s - \phi_t}}d\phi_s \tag{15}$$

Equation (15) allows us to derive an explicit general equation for the current, but the final expression [6] is somewhat cumbersome. Since both $Q_I'(\phi_s)$ and $dV_C/d\phi_s$ have been calculated from the same approximations, the output conductance corresponding to this choice satisfies Eq.(8), as stated by Baccarani et al. in [6].

In other words, a physically correct small-signal model is obtained from the approximation of Pao-Sah integral by choosing a consistent pair of input voltage and output current equations.

3.2 Classical strong inversion models

Classical strong inversion models may be derived from the Pao-Sah integral (2) by expressing the inversion charge density as a function of the channel voltage. According to this approach, the consistency test of Eq.(4) is thoroughly satisfied. For instance, a well-known dependence between inversion charge density and voltages [5] in strong inversion is given below

$$Q_I' = -C_{ox}'(V_G - V_{T0} - nV_C) \tag{16}$$

where n is the slope factor and V_{T0} is the equilibrium threshold voltage.

Substituting Eq. (16) into Eq. (1) and assuming n to be independent of the channel voltage V_C, it follows that

$$I_D = \mu C_{ox}'\frac{W}{L}\left[V_G - V_{T0} - \frac{n}{2}(V_S + V_D)\right](V_D - V_S) \tag{17}$$

which is valid in the so-called triode region. For n=1 (neglecting the body effect) Eq. (17) reduces to the classical textbook expression [5].

The output conductance of a transistor modeled by (17) is, for slope factor independent of the channel voltage, given by

$$g_d = \left.\frac{\partial I_D}{\partial V_D}\right|_{V_G,V_S} = \mu C_{ox}'\frac{W}{L}(V_G - V_{T0} - nV_C) \tag{18}$$

which satisfies the consistency condition (4), as expected.

One can find in the technical literature some slightly different ways of presenting Eq. (16). In some models, e.g. [9] and [10], n is a function of the gate voltage only, while in others, e.g. [5] and [11], n is also a function of the channel voltage. The integration of (1) is possibly much complex if n is a function of V_C, and sometimes an approximation is rather applied, which leads to the failure of the consistence test.

Classical strong inversion models may also be derived by substituting the strong inversion approximation of the surface potential $\phi_s = 2\phi_F + V_C$ into a surface potential model such as Eqs.(9) and (10). Since according to this approximation $dV_C/d\phi_s = 1$, the consistency test will be satisfied only if the diffusion component of the drain current is negligible, because (1) and (9) are equivalent for this condition.

3.3 Charge control models

In charge control models the x-equation is a relationship between Q'_I and the potentials V_C and V_G. The y-equation is generally obtained by changing the integration variable from channel potential to inversion charge density in the Pao-Sah equation (2), as indicated below

$$I_D = -\mu \frac{W}{L} \int_{Q'_{IS}}^{Q'_{ID}} Q'_I \frac{dV_C}{dQ'_I} dQ'_I \tag{19}$$

Therefore, provided that the term dV_C/dQ'_I is calculated from the x-equation, it is straightforward that the consistency test here presented is always satisfied by charge control models.

For instance, the unified charge control model (UCCM), in differential form is

$$dQ'_I \left(\frac{1}{nC'_{ox}} - \frac{\phi_t}{Q'_I} \right) = dV_C \tag{20}$$

where $n = 1 + C'_b/C'_{ox}$, and C'_b is the depletion capacitance, calculated neglecting the carrier charge. In this case, n is a function of the gate voltage and does not depend on the drain or source voltages [9].

Substitution of (20) into (19) and integration from source to drain results in

$$I_D = \frac{\mu W}{L} \left(\frac{Q'_{IS} + Q'_{ID}}{2} - nC'_{ox}\phi_t \right) \left(\frac{Q'_{IS} - Q'_{ID}}{nC'_{ox}} \right) \tag{21}$$

By differentiating (21) with respect to V_D and calculating $\partial Q'_I / \partial V_D \big|_{V_G, V_S}$ through (20), Eq.(4) is obtained.

If, instead of (19), (9) is used to derive the current output equation of the charge control model, consistency may not be achieved. Nevertheless, this is not the case of UCCM which is based on the following approximations: linearity of the relationship between inversion charge density and surface potential

$$dQ'_I = nC'_{ox}\phi_s \tag{22}$$

and proportionality of the inversion capacitance with the inversion charge density

$$C'_i = -\frac{dQ'_I}{d\phi_s} = -\frac{Q'_I}{\phi_t} \tag{23}$$

Therefore, UCCM succeeds in conciliating (9) and Pao-Sah equation (1).

4 SUMMARY

Table I summarizes the analysis of compact models accomplished in Section 3, concerning the consistency test based on Pao-Sah equation. In most compact models the current output equation is either derived directly from Pao-Sah integral or from the charge-sheet approximation in (9). In the first case, the consistency test fails only if an approximation other than the voltage input equation is introduced into Pao-Sah integral in order to simplify its calculation. In the second case, the consistency test succeeds only if Pao-Sah equation (1) and Eq.(9) are conciliated by means of the following condition:

$$\frac{dQ'_I}{dV_C} = -\frac{Q'_I}{\phi_t} \left(1 - \frac{d\phi_s}{dV_C} \right) \tag{24}$$

The relevance of the use of the same approximations for both the input voltage equation and the output current equation of a MOSFET compact model has been presented. Inconsistent approximations in the input and output equations will lead to small-signal parameters which do not comply with the requirements of the physically ‘exact’ Pao-Sah model.

Model	x-equation	y-equation	consistency test
Brews' (surface potential)	$\left(V_G - V_{FB} - \phi_s\right)^2 =$ $\gamma^2\left(\phi_t e^{(\phi_s - 2\phi_F - V_C)/\phi_t} + \phi_S - \phi_t\right)$	$I_D = \mu \frac{W}{L} C'_{ox} \{(V_G - V_{FB})(\phi_{sL} - \phi_{s0}) - \frac{1}{2}(\phi_{sL}^2 - \phi_{s0}^2)$ $- \frac{2}{3}\gamma\left[(\phi_{sL} - \phi_t)^{3/2} - (\phi_{s0} - \phi_t)^{3/2}\right] +$ $\phi_t(\phi_{sL} - \phi_{s0}) + \gamma\phi_t\left[(\phi_{sL} - \phi_t)^{1/2} - (\phi_{s0} - \phi_t)^{1/2}\right]\}$	Fails
Surface potential through approximation of Pao-Sah integral	$\left(V_G - V_{FB} - \phi_s\right)^2 =$ $\gamma^2\left(\phi_t e^{(\phi_s - 2\phi_F - V_C)/\phi_t} + \phi_S - \phi_t\right)$	$I_D = -\mu \frac{W}{L} \int_{\phi_{s0}}^{\phi_{sL}} Q'_I(\phi_s) \frac{dV_C}{d\phi_s} d\phi_s$	Depends upon $\frac{dV_C}{d\phi_s}$
Classical strong inversion	$Q'_I = -C'_{ox}\left(V_G - V_{T0} - nV_C\right)$	$I_D = \mu C'_{ox} \frac{W}{L}\left[V_G - V_{T0} - \frac{n}{2}(V_S + V_D)\right](V_D - V_S)$	Succeeds if $n=n(V_G)$
UCCM	$dQ'_I\left(\frac{1}{nC'_{ox}} - \frac{\phi_t}{Q'_I}\right) = dV_C$	$I_D = \frac{\mu W}{L}\left(\frac{Q'_{IS} + Q'_{ID}}{2} - nC'_{ox}\phi_t\right)\left(\frac{Q'_{IS} - Q'_{ID}}{nC'_{ox}}\right)$	Succeeds if $n=n(V_G)$

Table I: Consistency test applied to a sample of compact models

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